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Extended Dualization: a method for the Bosonization of Anomalous Fermion Systems in Arbitrary Dimension

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Abstract

The technique of extended dualization developed in this paper is used to bosonize quantized fermion systems in arbitrary dimension D in the low energy regime. In its original (minimal) form, dualization is restricted to models wherein it is possible to define a dynamical quantized conserved charge. We generalize the usual dualization prescription to include systems with dynamical non–conserved quantum currents.

Bosonization based on this extended dualization requires the introduction of an additional rank 0 (scalar) field together with the usual antisymmetric tensor field of rank $(D-2)$. Our generalized dualization prescription permits one to clearly distinguish the arbitrariness in the bosonization from the arbitrariness in the quantization of the system.

We study the bosonization of four–fermion interactions with large mass in arbitrary dimension. First, we observe that dualization permits one to formally bosonize these models by invoking the bosonization of the free massive Dirac fermion and adding some extra model–dependent bosonic terms.

Secondly, we explore the potential of extended dualization by considering the particular case of chiral four–fermion interactions. Here minimal dualization is inadequate for calculating the extra bosonic terms. We demonstrate the utility of extended dualization by successfully completing the bosonization of this chiral model.

Finally, we consider two examples in two dimensions which illuminate the utility of using extended dualization by showing how quantization ambiguities in a fermionic theory propagate into the bosonized version. An explicit parametrization of the quantization ambiguities of the chiral current in the Chiral Schwinger model is obtained. Similarly, for the sine–Gordon interaction in the massive Thirring model the quantization arbitrariness is explicitly exhibited and parametrized.

I. INTRODUCTION

Bosonization is a procedure for converting a given fermion field theory into its bosonic equivalent. This equivalence is to be understood at the quantum level, as an equivalence between the Green functions of the two quantum theories. The importance of bosonization is clear, since it permits one to investigate quantum fermion systems by using bosonic techniques, which are always more powerful and better developed.

The bosonization program started a considerable time ago, but for many years the known procedures were very restrictive, only applying to theories in two dimensional ($D = 2$) spacetime [1–4]. More recently there has been a lot of interest in extending bosonization to higher dimensions [5].

A new direction of inquiry has been recently opened by applying dualization techniques to the purpose of bosonization [6–9]. Dualization can be applied to arbitrary spacetime dimension D, and also to arbitrary fermion field theories, provided that a conserved quantum charge can be defined. We will call this original prescription "minimal dualization".

Given a D–dimensional fermion system with a dynamical conserved quantum current, minimal dualization guarantees the existence of a bosonized version. The explicit bosonic action is obtained by integrating out an auxiliary gauge field. The bosonic variable is an antisymmetric tensor field of rank $D-2$, a $(D-2)$–form, denoted $\Lambda^{(D-2)}$.

Minimal dualization rewrites the conserved quantized current density $\mathcal{J}^{(1)}$, in terms of the real field $\Lambda^{(D-2)}$ as

$$\mathcal{J}^{(1)} = *d\Lambda^{(D-2)}. \tag{1.1}$$

Here d is the exterior differential and $*$ the Hodge star operation.

Alternatively, in explicit components,

$$\mathcal{J}_\mu(x) = \epsilon_{\mu\mu_1\ldots\mu_{D-1}}\partial^{\mu_1}\Lambda^{\mu_2\ldots\mu_{D-1}}(x). \tag{1.2}$$

For $D = 2$ minimal dualization [6,8] gives the same results as conventional bosonization [10].

In this paper we will extend the minimal dualization procedure to include anomalous quantum fermion systems. We define an anomalous system as a system with a non–conserved quantum dynamical current. In that case, the current density $\mathcal{J}^{(1)}$ is not necessarily conserved.

In extended dualization, the bosonic equivalent action depends on the previously introduced antisymmetric rank $(D-2)$ field, plus a rank 0 real scalar field $\lambda^{(0)}$. The relation between the current density and the bosonic fields is modified to

$$i\mathcal{J}^{(1)} = d\lambda^{(0)} + i{*}d\Lambda^{(D-2)}. \tag{1.3}$$

If the quantization of the fermion fields is compatible with the conservation of the current $\mathcal{J}^{(1)}$, we will see that the scalar field λ is equal to zero and we recover the usual minimal dualization.

Both (1.1) and (1.3) imply that for $D > 2$, the tensor field $\Lambda^{(D-2)}$ is not uniquely defined. If two $(D-2)$–forms Λ and Λ' are related by

$$\Lambda' - \Lambda = d\chi^{(D-3)}, \tag{1.4}$$

for some antisymmetric $(D-3)$-form, $\chi^{(D-3)}$, then these two $(D-2)$–forms yield the same current. The bosonic action has, by construction, a gauge symmetry. The $(D-2)$–form Λ is a so–called "gauge form" (generalization of a gauge field) [8].

In Section 2, we will give a full description of the extended dualization method, and show that it permits the bosonization of anomalous fermion theories in arbitrary dimension.

Extended dualization is a generalization of minimal dualization to include anomalous quantum systems. It also exhibits other virtues that deserve mention here. Within our extended approach, we can reinterpret minimal dualization in a very simple way. This reinterpretation clearly shows that there is a considerable amount of freedom involved in the

dualization procedure. This arbitrariness in the bosonization is clearly differentiated from the arbitrariness in the quantization of the fermion system.

Although minimal dualization allows one to (at least in principle) bosonize any arbitrary fermion theory with a dynamical conserved current, it does not guarantee the corresponding bosonic action to be easily tractable or even local. The use of the above mentioned arbitrariness in dualization, that we identify in this paper, can be very convenient in order to find the most tractable form of bosonization. For instance, we know that systems of non–relativistic fermions at positive density yield a well behaved bosonic action using minimal dualization [8]. However, even for the relatively simple case of a free massive Dirac fermion, the mass makes dualization nontrivial.

The situation we will encounter in this paper is the following:

- For $D = 2$, dualization reproduces all the well known results of conventional bosonization [2,11,12]. The advantage of extended dualization is that it permits to bosonize the most general quantization of the fermionic system.

- For $D \geq 3$, we explore the low energy regime. The resulting bosonic action turns out to be local. By contrast, it is not possible to obtain an equivalent bosonic formulation via dualization in the high energy regime. It is in this restricted sense that one has bosonization in arbitrary dimension.

Dualization has the nice property that it permits one to obtain relationships among a wide class of models. We will apply this property to obtain the bosonic equivalent of the D–dimensional massive Thirring model (and other subsidiary models) very simply, by adding some extra model–dependent bosonic terms to the bosonization of the massive Dirac fermion. We remark that, in contradistinction to minimal dualization, extended dualization allows one to bosonize more general four–fermion interaction models including **chiral** currents (which are, in general, anomalous).

The dimension $D = 2$ is a special case that we consider separately in Section 2. As an example we discuss the application of extended dualization to the bosonization of the Thirring

model. We obtain the explicit dependence of the sine–Gordon equivalent interaction and the bosonic version of the currents on the parametrization of the quantization arbitrariness.

Section 3 will deal with the low energy regime, in arbitrary dimension D. For large fermion mass, we study the bosonization of massive four–fermion interaction models. The case of chiral four–fermion interactions is particularly interesting because it can only be treated within the framework of extended dualization.

One should mention here that some other work has been done in $D > 2$ bosonization, for non–anomalous systems, using frameworks different from dualization. In particular, for the massive Thirring model in dimension $D = 3$, the results in the abelian case [13] are a particular case of the bosonic action obtained in this paper using extended dualization. On the other hand, the methods described in [13], and their extension to the non–abelian case [14], are quite convoluted. More importantly, those methods are very much tied to the peculiarities of the Thirring model. The dualization prescription, by contrast, is a nice simple technique that allows one to study a wide range of quantum fermion systems, among which the Thirring model is just a particularly simple example. Moreover, extended dualization is not restricted by any symmetry requirement and it also allows the bosonization of anomalous systems. In section 3 we successfully complete the bosonization of the chiral four–fermion interaction model.

Section 4 is devoted to the dualization of the simplest model in 2D without non–anomalous quantization, the Chiral Schwinger model. Extended dualization is very useful to show how quantization ambiguities affect the bosonized version of the model. An explicit parametrization of the quantization ambiguities associated with the Chiral Schingwer model is exhibited. In the bosonic version of the theory this quantization ambiguity manifests itself as an additional scalar. Moroever extended dualization permits one to write easily an explicit expression for the bosonic equivalent of the chiral current.

We will conclude, in Section 5, with a brief summary of the main results on extended dualization.

II. EXTENDED DUALIZATION: BOSONIZATION

A. Extended dualization prescription

Consider a quantum system of fermions in a flat euclidean spacetime* of arbitrary dimension D. The partition function is given by

$$Z = \int \mathcal{D}\psi \mathcal{D}\overline{\psi} \quad e^{-S_f(\psi,\overline{\psi},\phi)}, \tag{2.1}$$

where S_f includes any fermion self–interaction, and interactions with other external fields or source terms (generically denoted by ϕ).

We will start by describing a trick to modify the partition function (2.1). It consists of the introduction of a path–integral representation of the identity into the partition function. We call this trick "extended dualization". Later we will establish the relationship between such manipulations and the bosonization of the fermion system.

Consider the following integral representation of the identity:

$$1 = \int \mathcal{D}A\mathcal{D}[\Lambda]\mathcal{D}\lambda \quad e^{F(A,\psi,\overline{\psi},\phi)} e^{(A,d\lambda + i*d\Lambda)}. \tag{2.2}$$

Here $A^{(1)}$ is an auxiliary 1–form and the inner scalar product of forms† has been used.

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* In a general curved spacetime with non–trivial topology, the extended dualization procedure has to be modified, as it is indicated in appendix A

†Given two rank k–forms $a^{(k)} = \frac{1}{k!} a_{\mu_1 \dots \mu_k}(x) dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k}$, and $b^{(k)}$, we define the inner scalar product

$$(a, b) \equiv \int d^D x \quad a_{\mu_1 \dots \mu_k}(x) b^{\mu_1 \dots \mu_k}(x).$$

Then

$$(A, d\lambda) = \int d^D x A^\mu(x) \partial_\mu \lambda(x), \quad (A, i*d\Lambda) = i\epsilon_{\mu\mu_1 \dots \mu_{D-1}} \int d^D x A^\mu(x) \partial^{\mu_1} \Lambda^{\mu_2 \dots \mu_{D-1}}(x).$$

$F(A,\psi,\overline{\psi},\phi)$ is an arbitrary scalar functional depending on the original fields and the non–physical vector field $A^{(1)}$, subject to the condition

$$F(A=0,\psi,\overline{\psi},\phi)=0. \tag{2.3}$$

Furthermore, $\Lambda^{(D-2)}$ is an antisymmetric rank $(D-2)$ tensor field and $\lambda^{(0)}$ is a rank 0 scalar field. $\mathcal{D}[\Lambda]$ denotes the measure on the space of gauge orbits

$$[\Lambda]=\{\Lambda' \colon \Lambda'-\Lambda=d\chi^{(D-3)}\}. \tag{2.4}$$

We relegate the proof of (2.2) to Appendix A.

Now introduce the identity, in its path–integral form (2.2), into the fermion partition function (2.1). So far we have nothing but an equivalence among path–integral expressions,

$$\int \mathcal{D}\psi\mathcal{D}\overline{\psi} \quad e^{-S_f}=\int \mathcal{D}\psi\mathcal{D}\overline{\psi}\mathcal{D}A\mathcal{D}[\Lambda]\mathcal{D}\lambda \quad e^{-S_f+F(A,\psi,\overline{\psi},\phi)+(A,d\lambda+i*d\Lambda)}. \tag{2.5}$$

Now we want to extract from this identity a bosonic system in terms of which we could calculate Green functions of the fermionic theory. For the purpose of bosonization we will first suppose that the change in the order of integration leaves results unaltered. In the right hand side of (2.5) we will try to perform the integration over the fermions and the auxiliary field A, and leave an expression in terms of Λ and λ (which will be the fields of the bosonized action).

Define the bosonic action S_b,

$$e^{-S_b(\Lambda,\lambda,\phi)}\equiv\int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\overline{\psi} \quad e^{-S_f+F(A,\psi,\overline{\psi},\phi)+(A,d\lambda+i*d\Lambda)}. \tag{2.6}$$

By construction, we have

$$Z=\int \mathcal{D}\psi\mathcal{D}\overline{\psi} \quad e^{-S_f(\psi,\overline{\psi},\phi)}=\int \mathcal{D}[\Lambda]\mathcal{D}\lambda \quad e^{-S_b(\Lambda,\lambda,\phi)}. \tag{2.7}$$

Therefore, for any fermionic action $S_f(\psi,\overline{\psi},\phi)$, we have introduced an equivalent bosonic action $S_b(\Lambda,\lambda,\phi)$. The partition function of the quantum system, originally given in terms

of the fermionic field ψ, admits a bosonic representation in terms of the fields $\Lambda^{(D-2)}$ and $\lambda^{(0)}$.

It immediately follows from its definition that the bosonic action is invariant under an abelian gauge transformation of the form $\Lambda \to \Lambda + d\chi^{(D-3)}$. One says that the form $\Lambda^{(D-2)}$ is a gauge form, which is to be integrated over the space of gauge orbits.

So far, the bosonic action $S_b(\Lambda, \lambda, \phi)$ defined in (2.6) includes an arbitrary functional $F(A, \psi, \overline{\psi}, \phi)$. It is a matter of delicacy and judgement to choose F in such a way that the bosonic action, and the identification between Green functions of the fermion and boson models, become tractable.

In the case of a fermion system whose interaction involves a single current density $\mathcal{J}^{(1)}(\psi, \overline{\psi})$, there is a particularly natural and economic choice of the functional $F(A, \psi, \overline{\psi}, \phi)$:

$$F(A, \psi, \overline{\psi}, \phi) = -i(A, \mathcal{J}). \tag{2.8}$$

The convenience of the choice (2.8) is clear if, in (2.6), one first integrates out the auxiliary field A. The current then has a nice and simple bosonic equivalent

$$\int \mathcal{D}A \quad e^{-i(A,\mathcal{J})} e^{(A, d\lambda + i*d\Lambda)} = \delta(-i\mathcal{J} + d\lambda + i*d\Lambda). \tag{2.9}$$

Therefore, the bosonic action defined by

$$e^{-S_b(\Lambda,\lambda,\phi)} \equiv \int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\overline{\psi} \quad e^{-S_f - i(A,\mathcal{J}) + (A, d\lambda + i*d\Lambda)}, \tag{2.10}$$

makes the identification of the bosonic equivalent of the fermion current density very simple

$$i\mathcal{J}(x) = d\lambda(x) + i*d\Lambda(x). \tag{2.11}$$

If the dynamics of the system involves several current densities $\mathcal{J}^a$, the prescription (2.8) can be generalized by introducing as many auxiliary vector fields A^a, and bosonic fields λ^a and Λ^a, as there are different dynamical currents. Then one has the identification (2.11) for each label a. For example, this is one way to get a dual version of non–abelian systems (such as fermions with non–abelian gauge interactions or non–abelian Thirring models).

Remark: Bosonization consists of establishing a relationship between fermionic and bosonic variables and, because of its generality, it cannot be a unique procedure. In this paper we explicitly demonstrate that behaviour. From the point of view of finding equivalent representations of the given partition function (2.7), other choices for the functional $F(A, \psi, \overline{\psi}, \phi)$ [different from (2.8)] are equally valid. This arbitrariness in dualization, which differs from the quantization arbitrariness [inherent in the fermionic integral in (2.6)], is related to the freedom in choosing the bosonic variables (more precisely, to the relationship between bosonic and fermionic variables).

This function F can be used to simplify the bosonic action by, for example, cancelling bothersome terms in the A integration. This could be interesting when the fermionic effective action is not quadratic in A (as is the case for $D > 2$ abelian systems beyond the low energy regime, and for D–dimensional non–abelian systems in any energy regime).

Even more, the function F can help us by providing a symmetry in the resulting bosonic action. For example, for $D = 3$ the field Λ is an abelian gauge field. However, for non–abelian fermionic systems this abelian gauge field can be converted into a non–abelian one by playing with the arbitrariness in F [15]. We will go back to this point later on.

The price to pay for such extra freedom to choose F is that the identification of the bosonic equivalent of the current density will be more complicated than (2.11). This means that we are choosing other bosonic variables not so trivially related to the fermionic variables, but better adapted to the bosonic version of the theory.

In this paper we will illustrate extended dualization in simple cases where it is not necessary to use a nontrivial choice of bosonic variables.

B. Properties of bosonization by extended dualization

Let us concentrate on the properties of abelian bosonization defined by relation (2.10).

Property B.1 *S_f and S_b related by (2.10), describe the same partition function,*

$$Z = \int \mathcal{D}\psi\mathcal{D}\overline{\psi} \quad e^{-S_f(\psi,\overline{\psi},\phi)} = \int \mathcal{D}[\Lambda]\mathcal{D}\lambda \quad e^{-S_b(\Lambda,\lambda,\phi)}. \tag{2.12}$$

This property follows by construction (assuming the order of integration can be exchanged).

The expression for the fermi fields in the bosonic theory will, in general, be very complicated [3,8]. However, fermion bilinears like $\mathcal{J}^{(1)}$ have simple expressions.

Property B.2 *Any correlation function of the current $i\mathcal{J}(x)$ in the fermionic theory is related to a correlation function of $[d\lambda(x) + i{*}d\Lambda(x)]$ in the bosonic model by*

$$\langle i\mathcal{J}^{\mu_1}(x_1)...i\mathcal{J}^{\mu_n}(x_n)\rangle_f = \langle [d\lambda + i{*}d\Lambda]^{\mu_1}(x_1)...[d\lambda + i{*}d\Lambda]^{\mu_n}(x_n)\rangle_b. \tag{2.13}$$

The proof is very straightforward by doing some path–integral manipulations and changing the order of integrals.

Property B.3 *Consider a fermion–boson dual description of a given quantum system, where the actions $S_f(\psi,\overline{\psi},\phi)$ and $S_b(\Lambda,\lambda,\phi)$ are related by extended dualization. Any modified system obtained from the original one by adding current interaction terms in the fermionic description $S'_f = S_f + S_{int}(i\mathcal{J})$ has a bosonic dual given by $S'_b = S_b + S_{int}(d\lambda + i{*}d\Lambda)$*

As a consequence of this property we can trivially dualize a large class of systems obtained from a given one, by adding suitable current density interaction terms.
Examples are,

- Current–current interaction terms, such as

$$S_{int} = +\frac{1}{2}\int d^Dxd^Dy\mathcal{J}_\mu(x)V_{\mu\nu}(x,y)\mathcal{J}_\nu(y) = +\frac{1}{2}(\mathcal{J}, V\mathcal{J}), \tag{2.14}$$

modify the original bosonized action S_b by the term

$$-\frac{1}{2}\left(d\lambda + i{*}d\Lambda, V[d\lambda + i{*}d\Lambda]\right). \tag{2.15}$$

Therefore, quartic interactions of fermions become quadratic in terms of the bosonic fields.

- Source terms for the currents, $j^{(1)}$, and interactions with abelian gauge fields $a^{(1)}$, of the form

$$S_{int} = (i\mathcal{J}, j + a), \tag{2.16}$$

lead in the bosonic action to

$$(d\lambda + i{*}d\Lambda, j + a). \tag{2.17}$$

Property B.4 *If the quantum field theory admits a quantized dynamical conserved fermion current density, then the scalar field $\lambda^{(0)}$ is equal to zero, and one recovers minimal dualization.*

Proof:

Let $\mathcal{J}^{(1)}(\psi, \overline{\psi})$ be a quantized dynamical conserved current density of the system, and let us use this current density to define the bosonized version of the theory:

$$e^{-S_b(\Lambda,\lambda)} = \int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\overline{\psi} \quad e^{-S_f - i(A,\mathcal{J}) + (A, d\lambda + i{*}d\Lambda)}. \tag{2.18}$$

Now define the "effective action"

$$e^{-\Gamma(A)} \equiv \int \mathcal{D}\psi\mathcal{D}\overline{\psi} \quad e^{-S_f - i(A,\mathcal{J})}. \tag{2.19}$$

First note that, with this dualization, the quantum current density is conserved iff the effective action is gauge invariant.

$$d{*}\mathcal{J}^{(1)} = 0 \quad \Longleftrightarrow \Gamma(A) = \Gamma(A + d\alpha). \tag{2.20}$$

Let us study the consequences of this gauge symmetry on the bosonic action,

$$e^{-S_b(\Lambda,\lambda)} = \int \mathcal{D}A \quad e^{-\Gamma(A) + (A, d\lambda + i{*}d\Lambda)}. \tag{2.21}$$

Any non–harmonic one-form $A^{(1)}$ can be cast into the form

$$A^{(1)} = d\zeta^{(0)} + i{*}d\varphi^{(D-2)}. \tag{2.22}$$

In terms of the fields $\zeta^{(0)}$ and $\varphi^{(D-2)}$ we have,

$$\begin{aligned} e^{-S_b(\Lambda,\lambda)} &\propto \int \mathcal{D}\zeta\mathcal{D}\varphi \quad e^{-\Gamma(\varphi)+(d\zeta+i*d\varphi,d\lambda+i*d\Lambda)} \\ &= \int \mathcal{D}\zeta \quad e^{(d\zeta,d\lambda)} \int \mathcal{D}\varphi \quad e^{-\Gamma(\varphi)-(d\varphi,d\Lambda)} \\ &\propto \delta(\lambda) \int \mathcal{D}\varphi \quad e^{-\Gamma(\varphi)-(d\varphi,d\Lambda)} \\ &\propto \delta(\lambda) \int \mathcal{D}[A] \quad e^{-\Gamma(A)+(A,i*d\Lambda)}. \end{aligned} \tag{2.23}$$

Here $\mathcal{D}[A]$ is the integral over the space of gauge orbits.

The scalar field $\lambda^{(0)}$ trivially disappears because it is set to zero by the delta function. We can use

$$e^{-S_b(\Lambda)} \equiv \int \mathcal{D}[A] \quad e^{-\Gamma(A)+(A,i*d\Lambda)}, \tag{2.24}$$

to describe the partition function, originally introduced in terms of fermions. This is the result of minimal dualization [6,8,9].

Remark: Note that the extended dualization presented in this paper, first enlarges minimal dualization because it permits the discussion of anomalous theories. Second, simplifies minimal dualization because it is introduced by an integral representation of the identity. And third, it allows to identify a big freedom in the dualization procedure (associated to the choice of the functional F). This arbitrariness is clearly differentiated from the possible arbitrariness in the path–integral quantization.

Another advantage of extended dualization is that, in contrast with minimal dualization, one does not need to use any symmetry of the fermion system, even if such symmetry does exist. This fact opens new possibilities for bosonizing fermionic theories.

Perhaps the most remarkable examples where we can exploit this advantage are non–abelian systems. The natural extension of minimal dualization to this case involves non–abelian gauge fields A^a and tensor fields Λ^a. This last field acts as a Lagrange multiplier for the condition $\mathcal{F}^a_{\mu\nu}(A) = 0$ and makes the gauge field trivial due to the non–abelian symmetry of

the system [7,15]. The problem is now, that, the coupling between A and Λ is not linear in A and therefore it is not possible an exact identification of the fermionic current like (1.2). Due to this fact one loses all the usual properties such as the direct relation between Green functions (property 2.2.2) or the simple rule to introduce interactions (property 2.2.3).

However, extended dualization of non–abelian systems does not require any modification because one does not need to implement any non–abelian symmetry in the theory. As we said previously, one can apply our dualization prescription (2.10) independently to each current density $\mathcal{J}^a$ with the choice (2.8) for F which is linear in A^a. Proceeding in this way, all the properties of bosonization by extended dualization remain in the non–abelian case. We should remark here that this prescription would lead to an independent abelian gauge symmetry for each field A^a, in contradistinction with the non–abelian symmetry of minimal dualization.

The above prescription leads to a bosonic action with an abelian gauge symmetry for each field Λ^a. For $D = 3$ Λ is a vector field; if we want for the bosonic action to have a non–abelian symmetry with Λ^a_μ as a gauge field, extended dualization is again needed. The coupling between Λ^a_μ and $\mathcal{F}^a_{\mu\nu}(A)$ is not gauge invariant and one needs to use the freedom in the function F to get such non–abelian symmetry. It has been shown [15] that the mixed Chern–Simons term

$$\frac{i}{2\pi}\int d^3x \quad \epsilon^{\mu\nu\rho}\Lambda^a_\mu \mathcal{F}^a_{\nu\rho}(A) - \frac{i}{24\pi}\int d^3x \quad \epsilon^{\mu\nu\rho} f_{abc} A^a_\mu A^b_\nu A^c_\rho, \tag{2.25}$$

changes by an integer multiple of $2\pi i$ under simultaneous non–abelian gauge transformations of Λ^a_μ and A^a_μ. That is, a choice of $F(A)$ including a term proportional to

$$i\int d^3x \quad \epsilon^{\mu\nu\rho} f_{abc} A^a_\mu A^b_\nu A^c_\rho, \tag{2.26}$$

cancells the gauge non–invariance of the coupling between Λ^a_μ and $\mathcal{F}^a_{\mu\nu}(A)$ (up to an irrelevant multiple of $2\pi i$). In this way, one gets a bosonic action with a non–abelian symmetry as we had in the fermionic theory. Of course, with this prescription, we pay the price of loosing the above mentioned properties of bosonization.

For simplicity only abelian examples will be treated in this paper. A detailed analysis of non–abelian systems will be the subject of another paper.

C. $D=2$ free massless Dirac fermion and related models

The conventional bosonization of the $D=2$ free Dirac fermion is well known [10]. It is obtained by using a quantization where the vector–like fermion current density $\overline{\psi}\gamma_\mu\psi$ is conserved. The result is that the free Dirac fermion in $D=2$ is equivalent to a free scalar boson Λ. The bosonized version of the vector–like fermion current is $\epsilon_{\mu\nu}\partial^\nu\Lambda$. Minimal dualization agrees with conventional bosonization [6,8]. However, using all the arbitrariness in the quantization, one gets a gauge non–invariant result for the fermionic path–integral [16,17]. Using extended dualization it is possible to obtain an explicit parametrization of the quantization arbitrariness in the bosonized version of the system.

The partition function is defined by

$$Z_{free}=\int \mathcal{D}\psi\mathcal{D}\overline{\psi} \quad e^{-S_f^{free}}. \tag{2.27}$$

Here $S_f^{free}(\psi,\overline{\psi})$ stands for the free Dirac action in flat euclidean spacetime[‡]

$$S_f^{free}(\psi,\overline{\psi})=-\overline{\psi}\gamma^\mu\partial_\mu\psi. \tag{2.28}$$

In $D=2$ the antisymmetric form $\Lambda^{(D-2)}$ reduces to a scalar field (a zero-form). The bosonic action obtained by dualization is given in terms of the two scalar fields Λ and λ by

$$\begin{aligned} e^{-S_b^{free}(\Lambda,\lambda)} &= \int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\overline{\psi} \quad e^{-S_f^{free}-i(A,\mathcal{J})+(A,d\lambda+i*d\Lambda)} \\ &= \int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\overline{\psi} \quad e^{\overline{\psi}\gamma^\mu(\partial_\mu-iA_\mu)\psi+(A,d\lambda+i*d\Lambda)}. \end{aligned} \tag{2.29}$$

[‡]Our conventions are:

$$\gamma_\mu^\dagger=\gamma_\mu, \quad \{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu\nu}, \quad \gamma_5^\dagger=\gamma_5.$$

In $D=2$ we use $\gamma_\mu\gamma_5=i\epsilon_{\mu\nu}\gamma^\nu$.

Here we have dualized using the vector-like current $\mathcal{J}_\mu(x) = \overline{\psi}(x)\gamma_\mu\psi(x)$.

We introduce the effective action,

$$e^{-\Gamma(A)} = \int \mathcal{D}\psi\mathcal{D}\overline{\psi} \quad e^{\overline{\psi}\gamma^\mu(\partial_\mu - iA_\mu)\psi}, \tag{2.30}$$

in terms of which the bosonic action reads

$$S_b^{free}(\Lambda, \lambda) = -\log \int \mathcal{D}A \quad e^{-\Gamma(A)+(A,d\lambda+i*d\Lambda)}. \tag{2.31}$$

The effective action $\Gamma(A)$ can be calculated exactly. However, we will use a symmetry argument to obtain its value. Consider a local and arbitrary transformation over the fermion fields

$$\begin{aligned} \psi &\longrightarrow e^{i(\alpha+\beta\gamma_5)}\psi, \\ \overline{\psi} &\longrightarrow \overline{\psi}e^{-i(\alpha-\beta\gamma_5)}. \end{aligned} \tag{2.32}$$

Under such transformation the effective action has the following property

$$\Gamma(A) = \Gamma(A - d\alpha + i * d\beta) - \log J(\alpha, \beta, A), \tag{2.33}$$

where $J(\alpha, \beta, A)$ is the finite Jacobian [16] corresponding to the fermion transformation (2.32).

The most general expression for this Jacobian can be read, for example, from [17]. It has the form

$$\log J(\alpha, \beta, A) = \frac{i}{\pi}\left[\xi(\beta, \epsilon_{\mu\nu}\partial_\mu A_\nu) + i\eta(\alpha, \partial_\mu A_\mu)\right] - \frac{1}{2\pi}\left[\xi(d\beta, d\beta) + \eta(d\alpha, d\alpha)\right]. \tag{2.34}$$

Here ξ and η are parameters introduced to describe the particular quantization employed. They satisfy the relation $\xi + \eta = 1$.

The particular case ($\eta = 0$, $\xi = 1$) corresponds to a gauge invariant quantization. This special case is equivalent to minimal dualization [6,8].

Property (2.33) implies that the bosonic action satisfies

$$S_b^{free}(\Lambda,\lambda) = S_b^{free}\left(\Lambda + \frac{\xi}{\pi}\beta, \lambda + \frac{\eta}{\pi}\alpha\right) - \frac{1}{2\pi}\left[\xi(d\beta,d\beta) + \eta(d\alpha,d\alpha)\right] - (d\beta,d\Lambda) - (d\alpha,d\lambda). \tag{2.35}$$

As a consequence:

$$\begin{aligned} S_b^{free}(\Lambda,\lambda) - \frac{\pi}{2\xi}(d\Lambda,d\Lambda) \text{is invariant under} \Lambda \to \Lambda + \frac{\xi}{\pi}\beta, \quad \lambda \to \lambda, \\ S_b^{free}(\Lambda,\lambda) - \frac{\pi}{2\eta}(d\lambda,d\lambda) \text{is invariant under} \lambda \to \lambda + \frac{\eta}{\pi}\alpha, \quad \Lambda \to \Lambda. \end{aligned} \tag{2.36}$$

Therefore we get the final result

$$S_b^{free}(\Lambda,\lambda) = \frac{\pi}{2\xi}(d\Lambda,d\Lambda) + \frac{\pi}{2\eta}(d\lambda,d\lambda) \tag{2.37}$$

Scaling to canonical variables, $\Lambda \to \sqrt{\frac{\xi}{\pi}}\Lambda$ and $\lambda \to \sqrt{\frac{\eta}{\pi}}\lambda$, we find that the free Dirac fermion in $D = 2$ bosonizes into two free scalars. By construction we have the following identification for the fermion currents

$$\begin{aligned} \overline{\psi}\gamma_\mu\psi &\longleftrightarrow \sqrt{\frac{\xi}{\pi}}\epsilon_{\mu\nu}\partial^\nu\Lambda - \sqrt{\frac{\eta}{\pi}}i\partial_\mu\lambda, \\ \overline{\psi}\gamma_\mu\gamma_5\psi &\longleftrightarrow \sqrt{\frac{\eta}{\pi}}\epsilon_{\mu\nu}\partial^\nu\lambda - \sqrt{\frac{\xi}{\pi}}i\partial_\mu\Lambda. \end{aligned} \tag{2.38}$$

The gauge invariant result corresponds to $\lambda = 0$ and $\xi = 1$.

Other related models can be trivially bosonized once we have expression (2.37) in hand. In particular, consider the fermionic action given by

$$S_f = S_f^{free} + (i\mathcal{J}, j) + \frac{g}{2}(\mathcal{J},\mathcal{J}), \tag{2.39}$$

with a source term $j^{(1)}$ for the current density, and a Thirring–like interaction of strength g. Using the general properties of extended dualization we have

$$\begin{aligned} S_b(\Lambda,\lambda) &= S_b^{free}(\Lambda,\lambda) + (j, d\lambda + i * d\Lambda) - \frac{g}{2}(d\lambda + i{*}d\Lambda, d\lambda + i{*}d\Lambda) \\ &= \frac{1}{2}\left(\frac{\pi}{\xi} + g\right)(d\Lambda,d\Lambda) + \frac{1}{2}\left(\frac{\pi}{\eta} - g\right)(d\lambda,d\lambda) + (j, d\lambda + i{*}d\Lambda). \end{aligned} \tag{2.40}$$

The net effect of the fermion self–interaction is a change of the kinetic coefficients for the scalar fields.

The stability of the Thirring model requires

$$-\frac{\pi}{\xi} < g < \frac{\pi}{\eta}. \tag{2.41}$$

This is a generalization of the usual condition $-\pi < g$. See [2].

In the range of stability, we rescale Λ and λ to canonical variables

$$\Lambda \to \frac{\beta}{2\pi}\Lambda, \qquad \lambda \to \frac{\gamma}{2\pi}\lambda, \tag{2.42}$$

where β and γ are constants defined by

$$\frac{4\pi}{\beta^2} = \frac{1}{\xi} + \frac{g}{\pi}, \qquad \frac{4\pi}{\gamma^2} = \frac{1}{\eta} - \frac{g}{\pi}. \tag{2.43}$$

In terms of the canonical variables we have

$$S_b(\Lambda, \lambda) = \frac{1}{2}(d\Lambda, d\Lambda) + \frac{1}{2}(d\lambda, d\lambda) + (j, \frac{\gamma}{2\pi}d\lambda + \frac{\beta}{2\pi}i{*}d\Lambda). \tag{2.44}$$

The identification of currents is

$$\begin{aligned} \overline{\psi}\gamma_\mu\psi &\longleftrightarrow \frac{\beta}{2\pi}\epsilon_{\mu\nu}\partial^\nu\Lambda - \frac{\gamma}{2\pi}i\partial_\mu\lambda, \\ \overline{\psi}\gamma_\mu\gamma_5\psi &\longleftrightarrow \frac{\gamma}{2\pi}\epsilon_{\mu\nu}\partial^\nu\lambda - \frac{\beta}{2\pi}i\partial_\mu\Lambda. \end{aligned} \tag{2.45}$$

The particular case ($\beta^2 = 4\pi\xi$, $\gamma^2 = 4\pi\eta$) corresponds to the free Dirac fermion.

D. $D = 2$ free massive Dirac fermion and related models

The partition function is defined by

$$Z_{free}(m) = \int \mathcal{D}\psi\mathcal{D}\overline{\psi} \quad e^{-S_f^{free}(m;\psi,\overline{\psi})}, \tag{2.46}$$

while the free massive Dirac action has the form

$$S_f^{free}(m; \psi, \overline{\psi}) = -[\overline{\psi}\gamma^\mu\partial_\mu\psi + m\overline{\psi}\psi]. \tag{2.47}$$

Consequently, the bosonic action for a free massive Dirac fermion has the following path integral expression

$$
\begin{aligned}
S_b^{free}(m;\Lambda,\lambda) &= \int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\overline{\psi} \quad e^{-S_f^{free}-i(A,\mathcal{J})+(A,d\lambda+i*d\Lambda)} \\
&= \int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\overline{\psi} \quad e^{\overline{\psi}\gamma^\mu(\partial_\mu-iA_\mu)\psi+m\overline{\psi}\psi+(A,d\lambda+i*d\Lambda)}.
\end{aligned}
\tag{2.48}
$$

Now, introduce the effective action,

$$
e^{-\Gamma_m(A)} = \int \mathcal{D}\psi\mathcal{D}\overline{\psi} \quad e^{\overline{\psi}\gamma^\mu(\partial_\mu-iA_\mu)\psi+m\overline{\psi}\psi},
\tag{2.49}
$$

in terms of which

$$
S_b^{free}(m;\Lambda,\lambda) = \int \mathcal{D}A \quad e^{-\Gamma_m(A)+(A,d\lambda+i*d\Lambda)}.
\tag{2.50}
$$

The bosonic action $S_b^{free}(m;\Lambda,\lambda)$ can be calculated by a power expansion in the fermionic mass. This procedure is well known in the $D=2$ bosonization folklore [11,12]. The main steps of the calculation are exhibited in Appendix B, and the final result is

$$
\begin{aligned}
S_b^{free}(m;\Lambda,\lambda) &= S_b^{free}(0;\Lambda,\lambda) - 2m\Lambda_{UV}\int d^2x \cos\left[\frac{2\pi}{\xi}\Lambda(x)\right] \\
&= \frac{\pi}{2\xi}(d\Lambda,d\Lambda) + \frac{\pi}{2\eta}(d\lambda,d\lambda) - 2m\Lambda_{UV}\int d^2x \cos\left[\frac{2\pi}{\xi}\Lambda(x)\right].
\end{aligned}
\tag{2.51}
$$

Here Λ_{UV} is an ultraviolet cutoff required to regularize the integral over the auxiliary field A.

Adding other interactions and sources to the bosonized free massive Dirac fermion is, as before, trivial. Take the fermionic action to be

$$
S_f = S_f^{free}(m;\overline{\psi},\psi) + (i\mathcal{J},j) + \frac{g}{2}(\mathcal{J},\mathcal{J}).
\tag{2.52}
$$

The corresponding bosonic action is given by

$$
\begin{aligned}
S_b(m;\Lambda,\lambda,j) = &\frac{1}{2}\left(\frac{\pi}{\xi}+g\right)(d\Lambda,d\Lambda) + \frac{1}{2}\left(\frac{\pi}{\eta}-g\right)(d\lambda,d\lambda) \\
&-2m\Lambda_{UV}\int d^2x \cos\left[\frac{2\pi}{\xi}\Lambda(x)\right] + (j,d\lambda+i*d\Lambda).
\end{aligned}
\tag{2.53}
$$

Rescaling to the canonical variables introduced in (2.42), (2.43), we have

$$
\begin{aligned}
S_b(m;\Lambda,\lambda,j) = {} & \frac{1}{2}(d\Lambda,d\Lambda)+\frac{1}{2}(d\lambda,d\lambda) \\
& -2m\Lambda_{UV}\int d^2x\cos\left[\frac{\beta}{\xi}\Lambda(x)\right]+(j,\frac{\gamma}{2\pi}d\lambda+\frac{\beta}{2\pi}i{*}d\Lambda).
\end{aligned}
\tag{2.54}
$$

The identification of the fermion current densities is not modified from the massless case (2.45), neither is the stability condition (2.41).

This is a generalization of the well known result [2,11] that the massive Thirring model is equivalent to the sine–Gordon model. In our analysis there is also an additional scalar field λ, that plays a role in the identification of the fermionic current densities

$$
\begin{aligned}
\overline{\psi}\gamma_\mu\psi &\longleftrightarrow \frac{\beta}{2\pi}\epsilon_{\mu\nu}\partial^\nu\Lambda-\frac{\gamma}{2\pi}i\partial_\mu\lambda, \\
\overline{\psi}\gamma_\mu\gamma_5\psi &\longleftrightarrow \frac{\gamma}{2\pi}\epsilon_{\mu\nu}\partial^\nu\lambda-\frac{\beta}{2\pi}i\partial_\mu\Lambda.
\end{aligned}
\tag{2.55}
$$

The parameter values $\beta^2=4\pi\xi$, and $\gamma^2=4\pi\eta$, correspond to the massive free Dirac fermion system. In particular, setting $\beta^2=4\pi$, and $\gamma^2=0$, corresponds to minimal dualization.

III. D–DIMENSIONAL MASSIVE FOUR–FERMION INTERACTIONS

In this section we will concentrate on the dualization of D–dimensional massive four–fermion interactions. First of all, we shall formally derive the bosonic action by invoking the bosonization of the free massive Dirac fermion and then incorporating extra bosonic terms. These extra terms will depend both on the model and on the particular dualization prescription. Later on, we shall review the results obtained for the massive Thirring model using minimal dualization in the low energy limit. Finally, we will explore the potential advantages of extended dualization by performing the bosonization of the chiral four–fermion interaction model (low energy only).

For definiteness

$$
S_f^{4f}(m;\psi,\overline{\psi})=S_f^{free}(m;\psi,\overline{\psi})+\frac{g}{2}(\mathcal{J},\mathcal{J}).
\tag{3.1}
$$

Here $S_f^{free}(m;\psi,\overline{\psi})$ is the free massive Dirac action defined in (2.47), while the density current $\mathcal{J}^{(1)}(x)$ is a rank–one fermion bilinear.

The quantum system is defined by the partition function. Including a source $j^{(1)}$ for the current, this reads

$$Z_{4f}(j) = \int \mathcal{D}\psi\mathcal{D}\overline{\psi} \quad e^{-S_f^{4f}-(i\mathcal{J},j)}. \tag{3.2}$$

The extended dualization of this model proceeds as follows

$$Z_{4f}(j) = \int \mathcal{D}[\Lambda]\mathcal{D}\lambda \quad e^{-S_b^{4f}(m;\Lambda,\lambda,j)}. \tag{3.3}$$

The bosonic action S_b^{4f} is given by

$$S_b^{4f}(m;\Lambda,\lambda,j) = S_b^{free}(m;\Lambda,\lambda) + \frac{g}{2}\left[(d\Lambda,d\Lambda)-(d\lambda,d\lambda)\right] + (j,d\lambda + i * d\Lambda). \tag{3.4}$$

Therefore, to obtain the bosonic representation of the massive four–fermion interaction model, the only calculation left to do is the evaluation of bosonic action for the free massive Dirac fermion $S_b^{free}(m;\Lambda,\lambda)$

$$S_b^{free}(m;\Lambda,\lambda) = \int \mathcal{D}A \quad e^{-\Gamma_m(A)+(A,d\lambda+i*d\Lambda)}. \tag{3.5}$$

In general, for arbitrary mass and dimension, the effective action $\Gamma_m(A)$, given by (2.49), is a very complicated functional. However, in the limit of large fermionic mass, $m\to\infty$, the effective action becomes quadratic in the field A and we can give an explicit expression for the bosonic action $S_b^{free}(m\nearrow\infty;\Lambda,\lambda)$.

A. Massive Thirring model: Low energy limit

The massive Thirring model is one of the simplest four–fermi interaction systems suitable for dualization. Other approaches have been used to bosonize the massive Thirring model [13]. However, dualization makes the bosonization a lot simpler. Additionally, while other approaches rely on rather specific properties of the Thirring model, dualization applies to any fermionic system.

In the Thirring model the current density, $\mathcal{J}_\mu(x) = \overline{\psi}\gamma_\mu\psi$ is vector–like. Therefore, the system admits a quantization that conserves the vector–like current and we can apply the minimal dualization prescription.

In the large mass limit, the gauge invariant effective action $\Gamma_m(A)$ is quadratic in the field A, and can be cast into the form

$$
\begin{aligned}
\Gamma_m(A) &= \frac{1}{2}\int d^D x \quad A^\mu(x) C^D_{\mu\nu}(\partial, m) A^\nu(x) \\
&= \frac{1}{2}(A, C^D A).
\end{aligned}
\tag{3.6}
$$

Using the well known results for the differential operator $C^D_{\mu\nu}(\partial, m)$ [9] , we can perform the gaussian integral in (3.5) in an appropiate gauge.

The final results for the Thirring model in the infinite mass limit are:

D=2 The bosonic action has a scalar field Λ with a mass that is proportional to the fermion mass and to the inverse of the Thirring coupling g. It is given by

$$
S_b^{(Th)}(m \nearrow \infty; \Lambda, j) = \frac{1}{2}\left[g + \frac{6\pi}{5}\right](d\Lambda, d\Lambda) + \frac{1}{2}6\pi m^2(\Lambda, \Lambda) + i(j, *d\Lambda). \tag{3.7}
$$

The mass for the scalar Λ is $M^2 = 5m^2/[1 + \frac{5g}{6\pi}]$.

D=3 The bosonic action includes a gauge field Λ_μ, with an abelian Chern–Simons term

$$
S_b^{(Th)}(m \nearrow \infty; \Lambda, j) = \frac{1}{2}g(d\Lambda, d\Lambda) + \frac{8\pi^2}{sign(m)}(\Lambda, *d\Lambda) + i(j, *d\Lambda). \tag{3.8}
$$

The gauge field Λ_μ has a topological mass [18] given by $M = 8\pi^2/g$.

D$\geq$ 4 The bosonic field is a rank $(D-2)$ antisymmetric form $\Lambda_{\mu_1 \ldots \mu_{D-2}}$, and the action becomes

$$
S_b^{(Th)}(m \nearrow \infty; \Lambda, j) = \frac{1}{2}g(d\Lambda, d\Lambda) + \frac{1}{K_D}(\Lambda, \Lambda) + i(j, *d\Lambda). \tag{3.9}
$$

Therefore, in terms of the gauge form $\Lambda^{(D-2)}$, the bosonic action is local in the infinite mass limit. The mass for the form Λ is $M^2 = 1/gK_D$, where K_D is a coefficient depending on the regularization method.

The interpretation of these results as an effective theory at low energies (much smaller than the fermion mass) will be discussed elsewhere [19].

B. Chiral four–fermion interactions: Low energy limit

A considerably less trivial system is obtained when the interaction is not parity invariant. For definiteness specialize the current in (3.1) to be $\mathcal{J}_\mu = \overline{\psi}\gamma_\mu \frac{1-\gamma_5}{2}\psi$. In this case, the relevant current of the interaction cannot be quantized in a conserved way. We therefore need to invoke extended dualization and thereby demonstrate its utility by successfully completing the bosonization of this chiral model.

The most general result for the effective action in the infinite mass limit is

$$\begin{aligned} \Gamma_m(A) &= -\log \int \mathcal{D}\psi\mathcal{D}\overline{\psi} \quad e^{\overline{\psi}\gamma^\mu(\partial_\mu - i\frac{1-\gamma_5}{2}A_\mu)\psi + m\overline{\psi}\psi}, \\ &= \frac{1}{2}\kappa_D(A,A), \end{aligned} \tag{3.10}$$

with a regularization dependent and dimension dependent coefficient κ_D.

The bosonic action is given by

$$\begin{aligned} S_b^{(chiral)}(m \nearrow \infty; \Lambda, \lambda, j) = &\frac{g}{2}\left[(d\Lambda, d\Lambda) - (d\lambda, d\lambda)\right] + i(j, d\lambda + i * d\Lambda) \\ &- \log \int \mathcal{D}A \quad e^{-\frac{1}{2}\kappa_D(A,A) + (A, d\lambda + i * d\Lambda)}, \end{aligned} \tag{3.11}$$

where $j^{(1)}$ is a source for the chiral current.

The bosonic action contains two fields $\Lambda^{(D-2)}$ and $\lambda^{(0)}$

$$S_b^{(chiral)}(m \nearrow \infty; \Lambda, \lambda, j) = \frac{1}{2}\left(g + \frac{1}{\kappa_D}\right)\left[(d\Lambda, d\Lambda) - (d\lambda, d\lambda)\right] + (j, d\lambda + i * d\Lambda). \tag{3.12}$$

This bosonic action is local and linear in the source, so we can establish the operator identification

$$i\overline{\psi}\gamma_\mu \frac{1-\gamma_5}{2}\psi \longleftrightarrow d\lambda + i * d\Lambda. \tag{3.13}$$

Because of the low energy limit, the bosonic action turns out to be bilinear in the bosonic fields. We also observe that the scalar λ has a kinetic term with the opposite sign. This

is a typical phenomenon related with anomalies that has already been encountered in the 2D quantization of chiral systems. (See for instance [20] and next section). One should expect the stability of the model to be based on some compensation of states coming from the scalar λ with some scalar states constructed with the tensor Λ.

IV. CHIRAL SCHWINGER MODEL: BOSONIZATION

The Chiral Schwinger model is the simplest model wherein the dynamical current is always non–conserved. Extended dualization permits to obtain the bosonic equivalent of the Chiral Schwinger model including a systematic treatment of the quantization arbitrariness. The explicit form of the chiral density current in terms of two scalar fields is obtained.

The partition function for the Chiral Schwinger model is defined by

$$Z_{CSM}(j) = \int \mathcal{D}a\mathcal{D}\psi\mathcal{D}\overline{\psi} \quad e^{-S_{CSM}(\psi,\overline{\psi},a)-i(j,\mathcal{J}^{ch})}. \tag{4.1}$$

The action has the form

$$S_{CSM}(\psi,\overline{\psi},a) = \frac{1}{4}(da,da) - \overline{\psi}\gamma^{\mu}\left[\partial_{\mu} - ie\frac{1-\gamma_5}{2}a_{\mu}\right]\psi. \tag{4.2}$$

Here $a^{(1)}$ stands for an abelian gauge field and the chiral current is

$$\mathcal{J}^{ch}_{\mu} = \overline{\psi}\gamma_{\mu}\frac{1-\gamma_5}{2}\psi. \tag{4.3}$$

As a result of integrating out the fermion fields we get an effective action

$$\Gamma_{CSM}(a) = -\log\int \mathcal{D}\psi\mathcal{D}\overline{\psi} \quad e^{\overline{\psi}\gamma^{\mu}\left[\partial_{\mu}-ie\frac{1-\gamma_5}{2}a_{\mu}\right]\psi}. \tag{4.4}$$

Evaluating this effective action, in its most general form, one finds [21,22]

$$\Gamma_{CSM}(a) = \frac{e^2}{2\pi}\left[\xi(a,a) - \left(a^{+}_{\mu}, \frac{\partial^{\mu}\partial^{\nu}}{\partial^2}a^{+}_{\nu}\right)\right]. \tag{4.5}$$

Here $a^{+}_{\mu} = \frac{1}{2}(\delta_{\mu\nu}+i\epsilon_{\mu\nu})a_{\nu}$. The arbitrariness in the quantization procedure is reflected in the presence of the local term (a,a), and this arbitrariness is parameterized by the coefficient ξ.

This is the result of conventional bosonization [21]. As a consequence of integrating out the fermi fields, the gauge field acquires a mass term, depending on the arbitrary parameter ξ. This effective action can be expressed in a local way by introducing an additional scalar field.

Bosonization via extended dualization goes beyond this conventional result since it permits one to exhibit the bosonic equivalent of the chiral current.

We see in (4.5) that the effective action is never gauge invariant for any value of ξ. Therefore the chiral current, $\mathcal{J}^{ch}$, is never conserved for any possible quantization of the Chiral Schwinger model. Extended dualization is the only option.

The partition function (4.1) can be expressed in terms of a bosonic action

$$Z_{CSM}(j) = \int \mathcal{D}a\mathcal{D}[\Lambda]\mathcal{D}\lambda \quad e^{-S_b^{CSM}(\Lambda,\lambda,a,j)}. \tag{4.6}$$

The bosonic action, using the extended dualization prescription, is given by

$$e^{-S_b^{CSM}(\Lambda,\lambda,a,j)} = \int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\overline{\psi} \quad e^{-S_{CSM}(\psi,\overline{\psi},a)-i(j,\mathcal{J}^{ch})-i(A,\mathcal{J}^{ch})+(A,d\lambda+i*d\Lambda)}. \tag{4.7}$$

After integrating over the fermions one gets

$$\begin{aligned} S_b^{CSM}(\Lambda,\lambda,a,j) = &+\frac{1}{4}(da,da) + (ea+j, d\lambda + i*d\Lambda) \\ &- \log \int \mathcal{D}A \quad e^{-\Gamma_{CSM}(A)+(A,d\lambda+i*d\Lambda)}. \end{aligned} \tag{4.8}$$

The gaussian integral over the auxiliary vector field A is straightforward and the result for the bosonic action is

$$\begin{aligned} S_b^{CSM}(\Lambda,\lambda,a,j) = &+\frac{1}{4}(da,da) + (ea+j, d\lambda + i*d\Lambda) \\ &+\frac{\pi(4\xi-1)}{8e^2\xi^2}(d\Lambda,d\Lambda) - \frac{\pi(4\xi+1)}{8e^2\xi^2}(d\lambda,d\lambda) - \frac{\pi}{4e^2\xi^2}(d\Lambda,d\lambda). \end{aligned} \tag{4.9}$$

Introducing canonical variables

$$\begin{aligned} \theta &= \frac{\sqrt{\pi}}{2e\xi}\left\{(1-2\xi)\Lambda + (1+2\xi)\lambda\right\}, \\ \phi &= \frac{\sqrt{\pi}}{e}(\Lambda - \lambda), \end{aligned} \tag{4.10}$$

the bosonic action becomes

$$\begin{aligned} S_b^{CSM}(\theta,\phi,a,j) = &+\frac{1}{4}(da,da) - \frac{1}{2}(d\theta,d\theta) + \frac{1}{2}(d\phi,d\phi) \\ &+\frac{e}{2\sqrt{\pi}}\left(ea+j, d\left[2\xi(\theta+\phi)-\phi\right]\right) \\ &+\frac{e}{2\sqrt{\pi}}\left(ea+j, i{*}d\left[2\xi(\theta+\phi)+\phi\right]\right). \end{aligned} \tag{4.11}$$

The bosonic action includes the gauge field $a^{(1)}$, plus two scalar fields θ and ϕ, coupled to this gauge field. The stability of the original model is based on a compensation between states coming from the scalar θ with some combination of the scalar ϕ and the (non–decoupled) zero component of the gauge field a [20].

The bosonic equivalent for the chiral current is given by

$$\begin{aligned} i\overline{\psi}\gamma_\mu\frac{1-\gamma_5}{2}\psi \longleftrightarrow &\frac{e}{2\sqrt{\pi}}\partial_\mu\left[2\xi(\theta+\phi)-\phi\right] \\ &+i\frac{e}{2\sqrt{\pi}}\epsilon_{\mu\nu}\partial^\nu\left[2\xi(\theta+\phi)+\phi\right]. \end{aligned} \tag{4.12}$$

The importance of this result is that bosonization based on extended dualization allows one to calculate correlation functions of the chiral current in the most general quantization of the Chiral Schwinger model using (4.12).[§]

V. CONCLUSIONS

We have introduced a constructive determination of the bosonic equivalent of a given anomalous fermi system, called extended dualization. The bosonic action includes both a scalar field and a rank $(D-2)$ antisymmetric form as fundamental fields. Our extended duality transformation is a generalization of minimal dualization. The last one applies only

[§]Recently, M. Garousi [23] has presented the dualization of the Chiral Schingwer model using only one scalar field. It is easy to see that his result corresponds to the particular case $\xi = 0$ in our scheme.

for systems in the presence of a dynamical conserved quantum charge and the only relevant bosonic field is the $(D-2)$–form.

We have seen that for a given fermionic system the bosonic counterpart is not unique. A large freedom is involved in the dualization transformation. One can choose the most convenient dualization in order to have the most tractable bosonic action and correlation functions for the relevant operators. In general, one can exploit this freedom to get specific properties for the bosonic action. In this paper we have explored one of the most simple options that is quite efficient for abelian bosonization when the fermionic effective action is quadratic.

A very nice property of dualization (either minimal or extended) is that it permits one to study a wide class of fermion systems by adding some extra model–related terms to the known bosonic version of some much simpler fermionic model. We have examined the bosonization of D–dimensional massive four–fermion interactions by adding appropriate terms to the bosonization of the D–dimensional free massive Dirac fermion.

We have demostrated the utility of extended dualization by explicitly exhibiting the bosonization of the particular case of the chiral four–fermion interaction model (in the low energy limit). The bosonization of this chiral model requires the inclusion of an additional scalar field in order to yield the bosonic equivalent of the chiral current density.

We have also applied the extended dualization procedure to determine the bosonized version of the Chiral Schwinger model and we have obtained the expression of the chiral current, for the most general quantization of the model, in terms of two independent scalar fields.

In all the above mentioned chiral models, extended dualization has permitted us to study easily the consequences of the quantization arbitrariness in the bosonic version of the system.

The considerable freedom in the extended dualization prescription introduced in this paper opens up the possibility of applying this freedom in order to dualize more complicated fermi systems (with or without anomalies), such as abelian systems in more than two dimensions beyond the low energy regime, and non–abelian systems. Moreover, this freedom

offers the possibility of imposing new symmetries on the dual action.

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Appendix A

In this appendix we will prove the identity

$$1 = \int \mathcal{D}A\mathcal{D}[\Lambda]\mathcal{D}\lambda \quad e^{F(A,\psi,\overline{\psi},\phi)} e^{(A, d\lambda + i*d\Lambda)}. \tag{1}$$

We assume that the fields are sufficiently well–behaved at the boundary that integration by parts is valid. Then:

- Integrating out the field Λ, we have

$$dA = 0. \tag{2}$$

- To integrate out the field λ we need to perform an analytic continuation $\lambda \to i\lambda$, and (at the end of the calculation) we need to return to the physical region of interest. This implies

$$d{*}A = 0. \tag{3}$$

Therefore, the result of integrating out the bosonic fields Λ and λ is

$$\int \mathcal{D}[\Lambda]\mathcal{D}\lambda \quad e^{(A, d\lambda + i*d\Lambda)} = \delta(dA)\delta(d{*}A), \tag{4}$$

so that the auxiliary vector field must be a harmonic one–form $A_h^{(1)}$. That is, $\Delta A_h = 0$, where Δ is the Laplacian that acts on one–forms in the D–dimensional space under consideration. Therefore

$$\int \mathcal{D}A\mathcal{D}[\Lambda]\mathcal{D}\lambda \quad e^{F(A,\psi,\overline{\psi},\phi)} e^{(A, d\lambda + i*d\Lambda)} = \int \mathcal{D}A_h e^{F(A_h,\psi,\overline{\psi},\phi)}. \tag{5}$$

In simple cases where spacetime has trivial topology, such as R^n or S^n, there are no harmonic one–forms, and therefore the identity (1) follows, provided only that

$$F(A = 0, \psi, \overline{\psi}, \phi) = 0. \tag{6}$$

In more complicated spacetimes with nontrivial topology, a modification of (1) is necessary, taking into account the space of harmonic one–forms. We generalize the path–integral representation of the identity in the following way: we replace the integration over the space of 1–forms $A^{(1)}$ by an integration over the space of orbits

$$[A] = \{A' : A' - A = A_h; \quad \Delta A_h = 0\}. \tag{7}$$

The generalization of (1) is obtained by taking a quotient over the space of harmonic one–forms, and then averaging the functional F over all harmonic one–form transformations,

$$1 = \int \mathcal{D}[A]\mathcal{D}[\Lambda]\mathcal{D}\lambda \quad e^{\int \mathcal{D}A_h F(A+A_h,\psi,\overline{\psi},\phi)} e^{(A,d\lambda+i*d\Lambda)}. \tag{8}$$

The proof is as follows: Integrating over the bosonic fields, we have

$$\int \mathcal{D}[A]\mathcal{D}[\Lambda]\mathcal{D}\lambda \quad e^{\int \mathcal{D}A_h F(A+A_h,\psi,\overline{\psi},\phi)} e^{(A,d\lambda+i*d\Lambda)} = e^{\int \mathcal{D}A_h F(A_h,\psi,\overline{\psi},\phi)}. \tag{9}$$

where translation invariance for the integration over harmonic 1–forms has been used. The identity (8) follows, provided that

$$\int \mathcal{D}A_h \; F(A_h,\psi,\overline{\psi},\phi) = 0. \tag{10}$$

This is the generalization, in the case of nontrivial harmonic one–forms, of the previous condition $F(A=0,\psi,\overline{\psi},\phi)a = 0$.

If the functional $F(A,\psi,\overline{\psi},\phi)$ is linear in the auxiliary vector field A, as it is the case for the simplest choice $F(A,\psi,\overline{\psi},\phi) = -i(A,\mathcal{J})$ [in (2.8)], the condition (10) is trivially satisfied and we have

$$\begin{aligned}\int \mathcal{D}A_h \; F(A+A_h,\psi,\overline{\psi},\phi) &= F(A,\psi,\overline{\psi},\phi)\int \mathcal{D}A_h + \\ &+ \int \mathcal{D}A_h \; F(A_h,\psi,\overline{\psi},\phi) = F(A,\psi,\overline{\psi},\phi)\end{aligned} \tag{11}$$

Therefore, in this case, the only new component to the dualization for a nontrivial topology comes from the explicit representation of the integration over harmonic orbits for the auxiliary vector field $A^{(1)}$ in (8). The simplest example, a two dimensional fermion in a cylinder, has been done in detail in [6].

Appendix B

This appendix is devoted to a sketch of the technical details of the $D = 2$ path integral calculation of

$$e^{-S_b^{free}(m;\Lambda,\lambda)} = \int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\overline{\psi} \quad e^{\overline{\psi}\gamma^\mu(\partial_\mu - iA_\mu)\psi + m\overline{\psi}\psi + (A, d\lambda + i*d\Lambda)}, \tag{12}$$

for small fermion mass, m.

We will proceed by performing a perturbative expansion in the mass, but first we rearrange this expression in a more convenient form.

Consider the following local transformation for the fermion fields:

$$\begin{aligned} \psi &\longrightarrow e^{i(\zeta+\varphi\gamma_5)}\psi, \\ \overline{\psi} &\longrightarrow \overline{\psi}e^{-i(\zeta-\varphi\gamma_5)}. \end{aligned} \tag{13}$$

The parameters of this transformation are the two scalar fields, ζ and φ, used to write the auxiliary vector field, $A = d\zeta + i*d\varphi$.

Under such a transformation we have

$$e^{-S_b^{free}(m;\Lambda,\lambda)} = \int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\overline{\psi} \quad J(A) \quad e^{\overline{\psi}\gamma^\mu\partial_\mu\psi + m\overline{\psi}e^{-2i\varphi\gamma_5}\psi + (A, d\lambda + i*d\Lambda)}. \tag{14}$$

The Jacobian $J(A)$ associated with the finite transformation (13) can be read off from the general result (2.34) in the particular case $\alpha = \zeta$, $\beta = -\varphi$. One finds

$$J(A) = \frac{1}{2\pi}\left[\xi(d\varphi, d\varphi) + \eta(d\zeta, d\zeta)\right]. \tag{15}$$

Because the mass term depends only on the field φ, the integration over ζ can be easily done. Apart from trivial constants we get

$$\begin{aligned} S_b^{free}(m;\Lambda,\lambda) = &\frac{\pi}{2\eta}(d\lambda, d\lambda) \\ &- \log \int \mathcal{D}\varphi\mathcal{D}\psi\mathcal{D}\overline{\psi} \quad e^{\overline{\psi}\gamma^\mu\partial_\mu\psi + m\overline{\psi}e^{-2i\varphi\gamma_5}\psi + \frac{\xi}{2\pi}(d\varphi, d\varphi) - (d\varphi, d\Lambda)}. \end{aligned} \tag{16}$$

Rescaling, to the canonical variable $\varphi \to \sqrt{\frac{\pi}{\xi}}\varphi + \frac{\pi}{\xi}\Lambda$, we have

$$S_b^{free}(m;\Lambda,\lambda) = \frac{\pi}{2\eta}(d\lambda,d\lambda) + \frac{\pi}{2\xi}(d\Lambda,d\Lambda) \qquad (17)$$

$$- \log \int \mathcal{D}\varphi \mathcal{D}\psi \mathcal{D}\overline{\psi} \quad e^{\overline{\psi}\gamma^\mu\partial_\mu\psi + m\overline{\psi}e^{-2i\left(\sqrt{\frac{\pi}{\xi}}\varphi+\frac{\pi}{\xi}\Lambda\right)\gamma_5}\psi + \frac{1}{2}(d\varphi,d\varphi)}.$$

Therefore

$$S_b^{free}(m;\Lambda,\lambda) = S_b^{free}(m=0;\Lambda,\lambda) \qquad (18)$$

$$- \log \int \mathcal{D}\varphi \quad e^{+\frac{1}{2}(d\varphi,d\varphi) - \Gamma_m[\sqrt{\frac{\pi}{\xi}}\varphi + \frac{\pi}{\xi}\Lambda]},$$

where the effective action Γ_m is the result of the fermionic integral

$$e^{-\Gamma_m[\Theta]} = \int \mathcal{D}\psi \mathcal{D}\overline{\psi} \quad e^{\overline{\psi}\gamma^\mu\partial_\mu\psi + m\overline{\psi}e^{-2i\Theta\gamma_5}\psi}. \qquad (19)$$

So far we have done nothing more than rearrange the integral in a convenient way. In order to perform (19) we will do an expansion in powers of the fermion mass.
Introducing $\sigma_\pm \equiv \overline{\psi}\frac{1}{2}(1\pm\gamma_5)\psi$, one gets

$$e^{-\Gamma_m[\Theta]} = \sum_{i=0}^{\infty} \frac{m^{2i}}{i!i!} \int \left[\prod_{k=1}^{i} d^2x_k d^2y_k\right] e^{2i\sum_k[\Theta(y_k)-\Theta(x_k)]} \qquad (20)$$

$$\int \mathcal{D}\psi \mathcal{D}\overline{\psi} \quad e^{\overline{\psi}\gamma^\mu\partial_\mu\psi} \prod_{k=1}^{i} \sigma_-(x_k)\sigma_+(y_k).$$

Using Zinn–Justin [11, p. 680, eq. (A28.14)], one sees

$$\int \mathcal{D}\psi \mathcal{D}\overline{\psi} e^{\overline{\psi}\gamma^\mu\partial_\mu\psi} \prod_{k=1}^{i} \sigma_-(x_k)\sigma_+(y_k) = \left(\frac{1}{2\pi}\right)^{2i} \frac{\prod_{k<l}^{i} |z_k - z_l|^2 |z'_k - z'_l|^2}{\prod_{k,l}^{i} |z_k - z'_l|^2}$$

$$\equiv K_i(x,y), \qquad (21)$$

where $z_k = x_k^0 + i x_k^1$, $z'_k = y_k^0 + i y_k^1$.

This implies

$$S_b^{free} = \frac{\pi}{2\eta}(d\lambda,d\lambda) + \frac{\pi}{2\xi}(d\Lambda,d\Lambda) \qquad (22)$$

$$- \log \sum_{i=0}^{\infty} \frac{m^{2i}}{i!i!} \int \left[\prod_{k=1}^{i} d^2x_k d^2y_k\right] K_i(x,y) e^{i\frac{2\pi}{\xi}\sum_k[\Lambda(y_k)-\Lambda(x_k)]}$$

$$\int \mathcal{D}\varphi \quad e^{+\frac{1}{2}(d\varphi,d\varphi)} \prod_{k=1}^{i} e^{2i\sqrt{\frac{\pi}{\xi}}\varphi(y_k)} \prod_{k=1}^{i} e^{-2i\sqrt{\frac{\pi}{\xi}}\varphi(x_k)}.$$

Now using Zinn–Justin [11, p. 664, eq. (28.13)],

$$\int \mathcal{D}\theta \quad e^{-\frac{1}{2t}(d\theta,d\theta)+i\sum_i \epsilon_i\theta(x_i)} \propto \begin{cases} 0 & \text{for } \sum_i \epsilon_i \neq 0 \\ \prod_{i<j}\left(\Lambda_{UV}|x_i-x_j|\right)^{\frac{\epsilon_i\epsilon_j t}{2\pi}} & \text{for } \sum_i \epsilon_i = 0 \end{cases} \tag{23}$$

for an ultraviolet cutoff Λ_{UV} that appears when one regularizes the free boson propagator. The fact that this correlation function is zero unless the coefficients satisfy the condition $\sum_i \epsilon_i = 0$ is a result of invariance under constant translations of the field θ.

The integral over the scalar fields φ results in

$$\int \mathcal{D}\varphi \quad e^{+\frac{1}{2}(d\varphi,d\varphi)} \quad \prod_{k=1}^{i} e^{2i\sqrt{\frac{\pi}{\xi}}\varphi(y_k)} \prod_{k=1}^{i} e^{-2i\sqrt{\frac{\pi}{\xi}}\varphi(x_k)} \propto (\Lambda_{UV})^{2i} K_i^{-1}(x,y). \tag{24}$$

The final result is

$$\begin{aligned} S_b^{free}(m;\Lambda,\lambda) &= \frac{\pi}{2\eta}(d\lambda,d\lambda) + \frac{\pi}{2\xi}(d\Lambda,d\Lambda) \\ &\quad - \log\sum_{i=0}^{\infty}\frac{(m\Lambda_{UV})^{2i}}{2i!}\begin{pmatrix} 2i \\ i \end{pmatrix}\left(\int d^2x e^{i\frac{2\pi}{\xi}\Lambda}\right)^i \left(\int d^2x e^{-i\frac{2\pi}{\xi}\Lambda}\right)^i. \end{aligned} \tag{25}$$

Consider now the partition function in the presence of a source j:

$$Z = \int \mathcal{D}[\Lambda]\mathcal{D}\lambda \quad e^{-S_b^{free}(m;\Lambda,\lambda)-(j,d\lambda+i*d\Lambda)}. \tag{26}$$

Invoking invariance under constant translations of the field Λ, we can show that the result of (23) for $\sum_i \epsilon_i \neq 0$ generalizes, in the presence of the source j, to

$$\begin{aligned} Z &= \int \mathcal{D}[\Lambda]\mathcal{D}\lambda \quad e^{-\frac{\pi}{2\eta}(d\lambda,d\lambda)-\frac{\pi}{2\xi}(d\Lambda,d\Lambda)-(j,d\lambda+i*d\Lambda)} \\ &\qquad \sum_{i=0}^{\infty}\frac{(m\Lambda_{UV})^{2i}}{2i!}\begin{pmatrix} 2i \\ i \end{pmatrix}\left(\int d^2x e^{i\frac{2\pi}{\xi}\Lambda}\right)^i \left(\int d^2x e^{-i\frac{2\pi}{\xi}\Lambda}\right)^i \\ &= \int \mathcal{D}[\Lambda]\mathcal{D}\lambda \quad e^{-\frac{\pi}{2\eta}(d\lambda,d\lambda)-\frac{\pi}{2\xi}(d\Lambda,d\Lambda)-(j,d\lambda+i*d\Lambda)} \\ &\qquad \sum_{i=0}^{\infty}\frac{(m\Lambda_{UV})^{2i}}{2i!}\sum_{k=0}^{2i}\begin{pmatrix} 2i \\ k \end{pmatrix}\left(\int d^2x e^{i\frac{2\pi}{\xi}\Lambda}\right)^k \left(\int d^2x e^{-i\frac{2\pi}{\xi}\Lambda}\right)^{2i-k} \\ &= \int \mathcal{D}[\Lambda]\mathcal{D}\lambda \quad e^{-\frac{\pi}{2\eta}(d\lambda,d\lambda)-\frac{\pi}{2\xi}(d\Lambda,d\Lambda)-(j,d\lambda+i*d\Lambda)} \\ &\qquad \sum_{i=0}^{\infty}\frac{(m\Lambda_{UV})^{2i}}{2i!}\left(\int d^2x e^{i\frac{2\pi}{\xi}\Lambda} + \int d^2x e^{-i\frac{2\pi}{\xi}\Lambda}\right)^{2i} \end{aligned} \tag{27}$$

$$
\begin{aligned}
&= \int \mathcal{D}[\Lambda]\mathcal{D}\lambda \quad e^{-\frac{\pi}{2\eta}(d\lambda,d\lambda)-\frac{\pi}{2\xi}(d\Lambda,d\Lambda)-(j,d\lambda+i*d\Lambda)} \\
&\qquad\qquad \sum_{i=0}^{\infty} \frac{(m\Lambda_{UV})^i}{i!} \left(\int d^2x e^{i\frac{2\pi}{\xi}\Lambda} + \int d^2x e^{-i\frac{2\pi}{\xi}\Lambda}\right)^i \\
&= \int \mathcal{D}[\Lambda]\mathcal{D}\lambda \quad e^{-\frac{\pi}{2\eta}(d\lambda,d\lambda)-\frac{\pi}{2\xi}(d\Lambda,d\Lambda)-(j,d\lambda+i*d\Lambda)} \\
&\qquad\qquad \exp\left(2m\Lambda_{UV}\int d^2x \cos\left[\frac{2\pi}{\xi}\Lambda\right]\right)
\end{aligned}
$$

This result is a generalization of the well known [2,3] equivalence of a massive Dirac fermion in $D = 2$ to the sine–Gordon theory, for the particular value $\xi = 1$.

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